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THE SUBGROUPS OF A TREE PRODUCT OF GROUPS[1]

BY

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ABSTRACT. Let $G = \Pi^*(A_i; U_{jk} = U_{kj})$ be a tree product with H a subgroup of G. By extending the technique of using a rewriting process we show that H is an HNN group whose base is a tree product with vertices of the form $xA_ix^{-1} \cap H$. The associated subgroups are contained in vertices of the base, and both the associated subgroups of H and the edges of its base are of the form $yU_{jk}y^{-1} \cap H$. The x and y are certain double coset representatives for $G \bmod (H, A_i)$ and $G \bmod (H, U_{jk})$, respectively, and the elements defined by the free part of H are specified. More precise information about H is given when H is either indecomposable or H satisfies a nontrivial law. Introducing direct tree products, we use our subgroup theorem to prove that if each edge of G is contained in the center of its two vertices then the cartesian subgoup of G is a free group. We also use our subgroup theorem in proving that if each edge of G is a finitely generated subgroup of finite index in both of its vertices and some edge is a proper subgroup of both its vertices then G is a finite extension of a free group iff the orders of the A_i are uniformly bounded.

1. Introduction. In 1958 the technique of using a rewriting process to investigate subgroups was used by Karrass and Solitar [8] to give proofs of the Nielsen-Schreier and Kuroš subgroup theorems. In 1970 the same authors [9] used the Kuroš rewriting process in their investigation of the subgroups of $(A * B; U)$. We continue the development of the rewriting process technique in extending the Karrass-Solitar subgroup theorem to arbitrary tree products.

If G is a tree product assign a level function to its graph having exactly one vertex of level zero and let G_n be the subtree product of the vertices of level $\leqslant n$. Then G_n is a tree product with amalgamations from the single vertex G_{n-1}, $n > 0$. If H is a subgroup of G and $H_n = H \cap G_n$ then H is the ascending union of the H_n. We use induction to find the structure of each H_n and show that H_n

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is naturally contained in H_{n+1}. Thus, the fundamental step in deriving our subgroup theorem is the case when G is a tree product with amalgamations from a single vertex. To treat this case we develop the compatible Kuroš rewriting process which not only distinguishes between generators from different vertices, but also between generators from different pairs of neighboring vertices. Theorem 5 gives the structure of H.

Indecomposable groups are those which do not admit proper decompositions into amalgamated products. Gaining information about the decomposition problem for tree products appears to be difficult since *an indecomposable subgroup of a tree product may be*: (i) *contained in a conjugate of a vertex*; (ii) *an ascending union of subgroups of conjugates of edges*; *or* (iii) *an HNN group with cyclic free part and with base generated by a pair of subgroups of conjugates of an edge* (Theorem 8). If a group G admits a proper decomposition $(A * B; U)$ then G cannot satisfy a nontrivial law unless $(A : U) = (B : U) = 2$. *The possible structures for a subgroup of a tree product which satisfies a nontrivial law are*: (i) *and* (ii) *above*; (iii) *above with the requirement that one of the pair of subgroups contain the other*; *and* (iv) *a subgroup of the form* $(A_H^x * B_H^y; U_H^z)$, where A and B are vertices of G, U is an edge, $z = x$ or y and X_H^g stands for $gXg^{-1} \cap H$ (Theorem 9).

A direct tree product is the quotient of a tree product whose edges are all contained in the centers of their vertices by its cartesian subgroup. Theorem 10 shows that *a direct tree product contains its vertices in the natural way and the subgroup generated by a subtree is its direct tree product.* As a consequence we generalize the well-known result that the cartesian subgroup of a free product is a free group as follows: *If G is a tree product such that each edge of G is contained in the center of both its vertices then the cartesian subgroup of G is a free group.* In [2] Anshel and Prener show that the commutator subgroup of a free product of finitely many finite abelian groups is a free group whose rank depends only on the number of factors and their orders. We show in Theorem 12 that *if G is a tree product of finitely many vertices each of which is a finite abelian group then the commutator subgroup of G is a free group whose rank depends only on the number of vertices and the orders of the vertices and edges.*

Karrass and Solitar [10] and Allenby and Gregorac [1] have proved that if $G = (A * B; U)$ with U a finitely generated proper subgroup of finite index in each factor then G is a finite extension of a free group iff A and B are both finite. To extend this result to tree products essentially all that need be changed is that the orders of the factors be uniformly bounded (see Theorem 13).

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2. **Notations.** We use standard notations for amalgamated products and tree products. In particular we adopt the following notations:

$\Pi^*(A_i;\ U_{jk} = U_{kj})$—the tree product with vertices A_i, edges U_{jk} and (implicit) amalgamating isomorphisms θ_{jk};

A_H^D—the subgroup $DAD^{-1} \cap H$;

$W \approx V$—the words W and V are freely equal;

x^ϵ—the symbol x raised to the power $\epsilon = \pm 1$;

${}^\delta(K)$—the δ-representative of K;

${}^\delta|K|$—the δ-double coset representative of K;

$G : (H, A)$—the number of double cosets in $G \bmod (H, A)$;

$|G|$—the order of the group G;

$C(G)$—the cartesian subgroup of G;

G'—the commutator subgroup of G;

S_K—the group of permutations on the elements of K.

3. **The compatible Kuroš rewriting process.** Let the generating symbols in a presentation for a group G be partitioned into subsets S_i and let each S_i be partitioned into two subsets α_i and β_i. Let H be a subgroup of G, and for each α_i and β_i select a right coset representative function for $G \bmod H$. We refer to the α_i- and β_i-representatives as S_i-representatives. For each S_i select a *locally neutral* right coset representative function $*_i$- for $G \bmod H$. Each of the $*_i$-representatives must be either an α_i- or a β_i-representative. Finally, a *neutral* right coset representative function $*$- for $G \bmod H$ is introduced, and we demand that each of the $*$-representatives be from some $*_i$-representative system.

If x is a generating symbol for G and N denotes a representative selected above then the following symbols are introduced:

(1) $s_{N,x}$ for $Nx^\delta(Nx)^{-1}$, where N and x are both α_i or β_i type and $\delta = \alpha_i$ or $\delta = \beta_i$ accordingly;

$t_{i,N}$ for $N({}^{*i}N)^{-1}$ where N is an S_i-representative; and

$r_{i,N}$ for $N({}^*N)^{-1}$ where N is a ${}^*i$-representative.

Let $W = Ux^\epsilon V$ be a word in the generating symbols of G which defines an element of H. If $x \in S_i$ we assign to x the ordered pair (i, δ), where $\delta = \alpha_i$ if $x \in \alpha_i$ and $\delta = \beta_i$ if $x \in \beta_i$. Now define τ to be a symbol-by-symbol replacement with x^ϵ replaced by

$$r^{-1}_{i,{}^{*i}(U)}\, t^{-1}_{i,{}^\delta(U)}\, s_{{}^\delta(U),x}\, t_{i,{}^\delta(Ux)}\, r_{i,{}^{*i}(Ux)} \quad \text{if } \epsilon = 1,$$

and

$$r^{-1}_{i,{}^*i(U)}\,t^{-1}_{i,\delta(U)}\,s^{-1}_{\delta(Ux),x}\,t_{i,\delta(Ux)}\,r_{i,{}^*i(Ux)} \quad \text{if } \epsilon = -1.$$

It is easy to calculate that the symbols replacing x^ϵ define $({}^*U)x[{}^*(Ux)]^{-1}$ if $\epsilon = 1$ (and $[{}^*(Ux^{-1})x({}^*U)^{-1}]^{-1}$ if $\epsilon = -1$) which are the generators arising from a Reidemeister rewriting process for G mod H on the neutral representatives. It follows that τ is a rewriting process and the symbols in (1) are generating symbols for H. We call τ a compatible Kuroš rewriting process. It is apparent from the definition of τ that if U_1 and U_2 are words in the generating symbols of G which define elements of H then $\tau(U_1) \approx \tau(U_2)$ if $U_1 \approx U_2$, $\tau(U_1U_2) = \tau(U_1)\tau(U_2)$ and $\tau(U_1^{-1}) = [\tau(U_1)]^{-1}$. By Theorem 2.6 of [11], a set of defining relations for H on the generating symbols in (1) is:

(2) $s_{N,x} = \tau(Nx[{}^\delta(Nx)]^{-1})$, with N and x of δ type where $\delta = \alpha_i$ or $\delta = \beta_i$ and $Nx \not\approx {}^\delta(Nx)$;

(3) $t_{i,N} = \tau(N[{}^{*i}N]^{-1})$, N of S_i type and $N \not\approx {}^{*i}N$;

(4) $r_{i,N} = \tau(N[{}^*N]^{-1})$, N a $*_i$-representative and $N \not\approx {}^*N$;

(5) $s_{N,x} = 1$ with N and x of δ type where $\delta = \alpha_i$ or $\delta = \beta_i$ and $Nx \approx {}^\delta(Nx)$;

(6) $t_{i,N} = 1$, N is of S_i type and $N \approx {}^{*i}N$;

(7) $r_{i,N} = 1$, N a $*_i$-representative and $N \approx {}^*N$;

(8) $\tau(KRK^{-1}) = 1$, where R runs through the set of defining relators of G and K runs through one of the representative systems.

We will now assume that all the representative systems are selected so that the extended Schreier property holds with respect to the α_i and β_i: That is, if $M = Nx^\epsilon$ is a representative then M and N are both δ-representatives if x is a δ symbol, $\delta =$ some α_i or β_i. We will use the extended Schreier property to eliminate (2)–(4) but first add the following relations to H:

(9) $\quad t_{i,N}r_{i,{}^*i_N} = t_{j,N}r_{j,{}^*j_N}$ for N both an S_i- and S_j-representative;

(10) $\quad r_{i,{}^*i_N} = t^{-1}_{i,{}^*N}$ for ${}^*N$ an S_i-representative.

(10) follows from (9) since ${}^*(N) = {}^{*k}(N)$ for some k: We include (10) for clarity.

LEMMA 1. *In computing $\tau(KRW^{-1})$ where K and W are S_i and S_j-representatives respectively, only the symbols $t_{i,K}r_{i,{}^*i_K}$ and $r^{-1}_{j,{}^*j_W}t^{-1}_{j,W}$ will remain from K and W^{-1}.*

PROOF. From the proof of the Kuroš subgroup theorem in [11] we know that the s symbols contributed by K are all of the type in (5). Moreover, if $K = Ux^{\epsilon 1}y^{\epsilon 2}V$ with x an (i, δ) generator and y a (j, ρ) generator then $y^{\epsilon 2}$ is replaced by symbols beginning with

$$r^{-1}_{j,\,{}^*j(Ux^{\epsilon 1})}t^{-1}_{j,\rho(Ux^{\epsilon 1})},$$

while the symbols replacing $x^{\epsilon 1}$ end with

$$t_{i,\delta(Ux^{\epsilon 1})}r_{i,\,{}^*i(Ux^{\epsilon 1})}.$$

By the extended Schreier property $Ux^{\epsilon 1}$ is both a δ- and ρ-representative so applying relation (9) whenever $i \neq j$ it is clear that all t and r symbols from K cancel except the first ones (which may be deleted since relations (6) and (7) apply) and the last ones which equal $t_{i,K}r_{i,\,{}^*iK}$ by (9).

Since $\tau(KRW^{-1}) = [\tau(WR^{-1}K^{-1})]^{-1}$ the only symbols remaining from W^{-1} will be $r^{-1}_{j,\,{}^*j(W)}t^{-1}_{j,W}$.

THEOREM 1. *Let H be a subgroup of G and let τ be a compatible Kuroš rewriting process for G* mod *H using an extended Schreier system. Then the symbols in* (1) *are generating symbols for H and the relations in* (5)–(9) *are a complete set of defining relations for H on these generating symbols.*

PROOF. We know the symbols in (1) are generating symbols for H and the relations in (2)–(9) are a set of defining relations for H on these generators.

Suppose that x is an (i, δ) generating symbol and N a δ-representative. Then $\tau(Nx[{}^\delta(Nx)]^{-1})$ is equivalent to

$$t_{i,N}r_{i,\,{}^*iN}[r^{-1}_{i,\,{}^*iN}t^{-1}_{i,N}s_{N,x}t_{i,\delta(Nx)}r_{i,\,{}^*i(Nx)}]r^{-1}_{i,\,{}^*i(Nx)}t^{-1}_{i,\delta(Nx)}.$$

It follows that the relations in (2) may be deleted.

If N is an S_i-representative then $\tau(N[{}^*iN]^{-1})$ is equivalent to

$$t_{i,N}r_{i,\,{}^*iN}r^{-1}_{i,\,{}^*iN}t^{-1}_{i,\,{}^*iN}$$

since ${}^*iN$ is an S_i-representative. It follows that the relations in (3) are superfluous in view of (6) and (9).

Finally, if N is a $*_i$-representative and ${}^*N$ is an S_j-representative then $\tau(N^*(N)^{-1})$ is equivalent to $t_{i,N}r_{i,N}r^{-1}_{j,\,{}^*iN}t^{-1}_{j,\,{}^*N}$. By relations (6) and (10) this word is equivalent to $r_{i,N}$ so the relations in (4) may be deleted.

4. The coset representative functions (cress) for the fundamental case. Let G be a tree product with amalgamations from the single vertex B and with any other vertex denoted by A_i. We assume each vertex is presented so that its generators contain a set of generators for each of its amalgamated subgroups. The amalgamation between A_i and B is denoted by $U_i = V_i$ and the corresponding defining relations for G are of the form $u_{ij} = v_{ij}$, where u_{ij} and v_{ij} are generators. This presentation of G is called *ordinary*.

We now fix a vertex A_1 and for each $i \neq 1$ let B_i be a copy of B presented the same as B but with the generating symbol b_{ij} of B_i corresponding to the generating symbol b_j of B. Present G by taking the union of our ordinary presentation of G and the presentations the B_i and adding all the relations $b_{ij} = b_j$. In this presentation G is a tree product with the simple path from A_i to A_1 being $(i \neq 1)$

$$A_i \underset{U_i=V_i}{*} B_i \underset{B_i=B}{*} B \underset{V_1=U_1}{*} A_1.$$

For notational convenience we use B_1 in place of B and generators b_{1j} for b_j. This presentation of G is called *large*. There is an obvious correspondence between ordinary and large presentations of G.

We now partition the generating symbols of a large presentation for G by letting S_i be the set of generating symbols for A_i and B_i. Let H be a subgroup of G and introduce α_i-, β_i-, $*_i$- and $*$-right coset representative functions as in the preceding section.

DEFINITION. Let G be given by a large presentation. A *cress* for G mod H consists of right coset representative functions α_i- and β_i-, one α_i- corresponding to each A_i and one β_i-corresponding to each B_i such that:

(i) The representative functions form a regular extended Schreier system for G mod H (where $*_i = \beta_i$ and $* = *_1 = \beta_1$).

(ii) When the u_i (v_i) symbols are deleted completely from the ends of the α_i- (β_i-) representatives the resulting words form a double coset representative system for G mod (H, U_i). We call these words the u_i- (v_i-) double coset representatives.

(iii) An α_i-representative does not end in a v_i symbol and a β_i-representative does not end in a u_i symbol.

(iv) If K is both a u_i- and v_i-double coset representative then $KP(u_i)$ is an α_i-representative iff $KP(v_i)$ is a β_i-representative.

(v) If an α_i-representative ends in an α_j symbol, $j \neq i$, then it is both an α_i- and β_i-double coset representative.

(vi) If W is any word in the generating symbols of G then ${}^{\beta_i}|W| = {}^{\beta_j}|W|$ for any i, j.

(vii) An S_i-representative cannot end in a β_j symbol, $i \neq j$.

DEFINITION. If W is a word in the generating symbols of G then $l_\delta(W)$ is the number of occurrences of a δ syllable among the syllables of W. The extremal length of W is $\Sigma l_{\alpha_i}(W)$.

We use $l(W)$ to stand for the extremal length of W.

THEOREM 2. *Let G be given by a large presentation. Then there is a cress for G* mod *H on the given generating symbols for G.*

PROOF. We will first construct a collection of coset representative systems $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$- using an ordinary presentation of G. There will be one $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$- corresponding to each U_i and V_i respectively. We then construct a cress from the $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$-. Let the $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$-representatives be written in the forms $D_iE_iP(u_i)$ and $\widetilde{D}_iF_iQ(v_i)$ respectively, where P, Q, E_iP and F_iQ are the maximal terminal segments consisting of u_i, v_i, α_i and β symbols. We will construct the representatives so that properties (a)–(d) below, which clearly imply properties (i)–(iii) and (v)–(vii) of a cress in the corresponding large presentation of G, are satisfied.

(a) The $\widetilde{D}_i$ form a double coset representative system for G mod (H, B) and each $\widetilde{D}_i$ is an α_j-representative ending in an α_j symbol. Moreover, $\{\widetilde{D}_i\} = \{\widetilde{D}_j\}$ for each β_j.

(b) For a fixed $\widetilde{D}_i$ the $\{F_iQ(v_i)\}$ adjoined to $\widetilde{D}_i$ to form $\widetilde{\beta}_i$-forms a special Schreier system for B mod $B \cap D_i^{-1}HD_i$ with respect to V_i.

(c) The D_i form a double coset representative system for G mod (H, A_i). Each D_i is a $\widetilde{\beta}_i$-representative ending in an F_i syllable (that is, a β syllable which is a terminal β syllable for some v_i-double coset representative) or an α_j-representative ending in an α_j symbol, $j \neq i$. In the latter case D_i is also a $\widetilde{\beta}_i$-double coset representative.

(d) For a fixed D_i the collection $\{E_iP(u_i)\}$ adjoined to D_i forms a special Schreier system for A_i mod $A_i \cap D_i^{-1}HD_i$ with respect to U_i.

We define the length of a double coset to be the minimum extremal length of a word in it. The construction will be to first select double coset representatives of minimal length in the double cosets they represent and supplement them with special Schreier systems (see [9] for the definition of a special Schreier system). We proceed by induction on the length n of the double cosets.

If $n = 0$ the only (H, B) coset of length n is HB. Select 1 as the representative of this double coset and for each V_j adjoin to 1 a special Schreier system for B mod $B \cap H$ with respect to V_j. If HWA_i has length 0 then W is a β syllable or $W = 1$. Then W is a word in HB so ${}^{\beta_i}(W)$ is defined and has the form $F_iQ(v_i)$. We let F_i be the double coset representative for HWA_i and adjoin to it a special Schreier system for A_i mod $A_i \cap F_i^{-1}HF_i$ with respect to U_i.

Assume we have selected $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$-representative for all cosets of H contained in an (H, B) or (H, A_i) coset of length less than r, $r > 0$.

Let HWB have length $r = l(W)$ and assume that W ends in an α_i symbol. Then HWA_i has length less than HWB so ${}^{\widetilde{\alpha}_i}(W)$ is defined and has the form $D_iE_iP(u_i)$. By inductive hypothesis $l(D_i) \leq r - 1$, and since D_iE_i is a word in HWB we must have $l(D_iE_i) \geq r$ so $E_i \neq 1$. Let $\widetilde{D}_i = D_iE_i$ be the double coset representative of HWB and for each V_j adjoin to $\widetilde{D}_i$ a special Schreier system for $B \bmod \widetilde{D}_i^{-1}H\widetilde{D}_i \cap B$ with respect to V_j.

To define double coset representatives for the double cosets HWA_i of length $r = l(W)$ notice that HWB has length $\leq r$. This means ${}^{\widetilde{\beta}_i}(W)$ is defined and has the form $\widetilde{D}_iF_iQ(v_j)$. Let $D_i = \widetilde{D}_iF_i$ be the double coset representative for HWA_i and adjoin to D_i a special Schreier system for $A_i \bmod D_i^{-1}HD_i \cap A_i$ with respect to U_i.

Whenever a word K is both a u_i- and v_i-double coset representative we may adjoin the same words in u_i and v_i symbols to K in constructing the $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$-representatives.

We have constructed the $\widetilde{\alpha}_i$- and the $\widetilde{\beta}_i$-(the *cresst*). Return now to the large presentation of G. We associate to each A_i and B_i the representative functions α_i- and β_i- which result when all the F_j and $Q(v_j)$ are replaced by the same words written in β_j symbols instead of β symbols. No ambiguity results since the F_j are indexed.

The systems $\widetilde{\alpha}_i$- and $\widetilde{\beta}_i$- are called a *cresst*, and we shall use this notion later.

5. The subgroup of a tree product with amalgamations from a single vertex. In this section G is a tree product with amalgamations from a single vertex. We continue the notations of §§3 and 4.

THEOREM 3. *Let H be a subgroup of G and let $\{D_iE_iP(u_i)\}$ and $\{DF_iQ(v_i)\}$ be the S_i-representatives in a cress for G* mod *H. Then H is an HNN group whose free part is freely generated by those t_{i,D_iE_i} such that D_iE_i is not an α_i- or β-double coset representative. Moreover, the base of H is a tree product whose vertices are all the subgroups $A_{iH}^{D_i}$ and B_H^D and the following hold*:

(1) *The neighboring vertices are all pairs $A_{iH}^{D_i}, B_H^D$ where $D_i = D$ or $D_i = DF_i$ or $D = D_iE_i$*;

(2) *The group amalgamated between neighbors $A_{iH}^{D_i}$ and B_H^D is $U_{iH}^{D_1}$ where $D_1 = D_i = D$ or D_1 is the longer of D_i and D in syllable length*;

(3) *The relations involving the associated subgroups and free part of H are given by*

$$t_{i,D_iE_i} U_{iH}^{DF_i} t_{i,D_iE_i}^{-1} = U_{iH}^{D_iE_i}$$

where DF_i is the v_i-double coset representative of D_iE_i.

PROOF. We may assume that G is given by a large presentation and that the cress is on the given generating symbols for G. We present H using the compatible Kuroš rewriting process and the given cress for G mod H. Then H has defining relations (5)–(9) as described by Theorem 1.

Consider the relations in (9). If N is both an S_i- and S_j-representative, $i \neq j$, then property (vii) of a cress implies that N must end in some α_r symbol. Further, N must be an α_i- or β_i-representative and we may assume that $r \neq i$. Now N must be a β-double coset representative, for if N is a β_i-representative then N is a β-double coset representative, while if N is an α_i-representative then by property (v) of a cress N is a β-double coset representative. From property (vi) it follows that the relations in (9) are superfluous in view of those in (6) and (7). Thus, (5)–(8) is a set of defining relations for H. The relations entering from (8) will be:

(8a) $\tau(KRK^{-1}) = 1$ where R runs through the defining relators from some A_i or B_i and K ranges over the α_i- or β_i-representative system accordingly;

(8b) $\tau(Ku_r v_{ir}^{-1} K^{-1}) = 1$ where u_r runs through the U_i generating symbols of A_i, v_{ir} runs through the V_i generating symbols of B_i and K ranges over the α_i-representatives;

(8c) $\tau(Kb_{ij} b_{1j}^{-1} K^{-1}) = 1$ where b_{ij} and b_{1j} run through the corresponding generating symbols of B_i and B_1, respectively, and K ranges over the β_i-representatives.

By Lemma 1 it is easy to calculate that the relations in (8a) may be regarded as involving only s_{M,a_i} or only $s_{L,b_{ij}}$ symbols. The proof of Theorem 5 of [9] assures that the s symbols and relations (5) and (8a) may be partitioned so that they form sets of generating symbols and defining relations for all the $A_{i_H}^{D_i}$ and $B_{i_H}^{D}$.

We now calculate the relations in (8b):

$$\tau(Ku_r v_{ir}^{-1} K^{-1}) = t_{i,K} r_{i,{}^{*}i_K} \cdot [r_{i,{}^{*}i_K}^{-1} t_{i,K}^{-1} s_{K,u_r} t_{i,{}^{\alpha}i(Ku_r)} \cdot r_{i,{}^{*}i(Ku_r)}]$$

$$\cdot [r_{i,{}^{*}i(Ku_r)}^{-1} t_{i,{}^{\beta}i(Ku_r)}^{-1} s_{{}^{\beta}i(K),v_{ir}}^{-1} t_{i,{}^{\beta}i(K)} r_{i,{}^{*}i(K)}] \cdot r_{i,{}^{*}i(K)}^{-1} t_{i,K}^{-1}$$

which reduces to $s_{K,u_r} t_{i,{}^{\alpha}i(Ku_r)} s_{{}^{\beta}i(K),v_{ir}}^{-1} t_{i,K}^{-1}$ since ${}^{*}i = {}^{\beta}i$. Thus, (8b) becomes

$$s_{K,u_r} = t_{i,K} s_{{}^{\beta}i(K),v_{ir}} t_{i,{}^{\alpha}i(Ku_r)}^{-1}. \tag{11}$$

If $K = D_i E_i Q$ and ${}^{\beta}i(K) = DF_i P$ then we may use the same calculations as in Theorem 5 of [9] to reduce (11) to

$$s_{K,u_r} = t_{i,D_iE_i} W(s_{{}^{\beta}i(DF_iL),v_{ij}}) t_{i,D_iE_i}^{-1}, \tag{12}$$

and if $t_{i,D_iE_i} = 1$ to reduce (12) to

(13) $$s_{D_iE_iQ(u_j),u_r} = s_{DF_iQ(v_{ij}),v_{ir}}.$$

By property (iii) of a cress the t_{i,D_iE_iQ} with $Q \neq 1$ do not occur in relations (6), and in the calculations referred to we see that the t_{i,D_iE_iQ} may be expressed in terms of s symbols and t_{i,D_iE_i}. Thus, we may delete all the t_{i,D_iE_iQ} with $Q \neq 1$.

To see which t symbols remaining are relations in (6) notice that $t_{i,K} = 1$ is always such a relation if K is a β_i-representative. T' e others are those t_{i,D_iE_i} for which $D_iE_i = DF_i$, but this happens iff $E_i = 1, F_i - 1$, or $E_i = F_i = 1$. Thus, t_{i,D_iE_i} is not a relator in (6) iff D_iE_i is not an α_i- or β-double coset representative. All other t symbols may be deleted from the presentation of H.

In (13) we fix D_iE_i and let $Q(u_j)$ vary over its Schreier system for U_i mod $U_i \cap E_i^{-1}D_i^{-1}HD_iE_i$ while u_r varies over the U_i generating symbols. The left side of (13) generates the subgroup $U_{iH}^{D_iE_i}$ of $A_{iH}^{D_i}$ and the right side generates the subgroup $V_{iH}^{D_iF}$ of B_{iH}^{D}. Moreover, the arguments in [9] show (12) reduces to

$$U_{iH}^{D_iE_i} = t_{i,D_iE_i}U_{iH}^{DF_i}t_{i,D_iE_i}^{-1}$$

where the mapping

$$s_{K,u_r} \longrightarrow W(s_{\beta i(DF_iL),v_{ij}})$$

arising from (12) is an isomorphism between subgroups of $A_{iH}^{D_i}$ and B_{iH}^{D}.

Factoring by the normal subgroup generated by all the s symbols shows that the t_{i,D_iE_i} are a free set of generators for a free subgroup of H, where D_iE_i is not an α_i- or β-double coset representative.

We now examine (8c): $\tau(Kb_{ij}b_{1j}^{-1}K^{-1})$ reduces to

$$s_{K,b_{ij}}r_{i,*i(Kb_{ij})}s^{-1}_{\beta 1(K),b_{1j}}r^{-1}_{i,K}$$

since any t or r_1 symbol is a relator in (6) or (7), so any relation in (8c) becomes

(14) $$s_{K,b_{ij}} = r_{i,K}s_{\beta 1(K),b_{1j}}r^{-1}_{i,*i(Kb_{ij})}$$

where the mapping

$$\Phi: s_{K,b_{ij}} \longrightarrow r_{i,K}s_{\beta 1(K),b_{1j}}r^{-1}_{i,*i(Kb_{ij})}$$

defines the identity on B_H^D in H. Now, if $K = DF_iQ(v_{ij})$ then ${}^*K = D^*(F_iQ(v_{ij}))$

and we let $F_i(b_{1j})$ be the word obtained from F_i by replacing each b_{ij} with b_{1j}. Then

$$r_{i,K} = \tau(K[{}^*K]^{-1}) = \tau\{DF_i(b_{1j})Q(v_{1j})^{\beta 1}[F_i(b_{1j})Q(v_{1j})]^{-1}D^{-1}\}$$
$$= W_1(S_{\beta 1(DL),b_{1j}}),$$

and similarly

$$r_{i,{}^*i(Kb_{ij})} = W_2(S_{\beta 1(DM),b_{1j}}).$$

This shows (14) may be replaced by

$$S_{k,b_{ij}} = W'(S_{\beta 1(DL),b_{1j}}). \tag{15}$$

Only relations (7) now involve r symbols. Using the above calculations we may delete $r_{i,K}$ from the presentation for H provided $r_{i,K}$ is replaced by a word W_1 in the generators of B^D_{1H} when $r_{i,K}$ appears in (7). Since a set of defining relations for B^D_{1H} is already present we may delete $W_1 = 1$ from our relations for H. It follows that we may delete all the r symbols from our presentation. Moreover, using (15) we may delete the generators and defining relations for B^D_{iH}, $i \neq 1$. Since Φ was the identity mapping of B^D_H in H the $V^{DF_i}_{iH}$ are replaced by the same subgroups of $B^D_{1H} = B^D_H$.

To complete the proof we show the $A^{D_i}_{iH}$ and B^D_H and amalgamated subgroups $U^{D_1}_{iH}$ generate their tree product as claimed. To see this notice that if D has syllable length > 0 then $D = D_jE_j$ and B^D_H is connected to $A^{D_j}_{jH}$ (in the linear graph Γ associated with the base of H) and to no other vertex whose superscript has shorter length than D. Also, any $A^{D_i}_{iH}$ is jointed to a unique B^D_H where the syllable length of D is less than or equal to the syllable length of D_i. It follows that Γ is a tree.

Note. Since a cress may be formed by "expanding" a cresst (as in the proof of Theorem 2) and a cresst will be what results when the β_i symbols of a cress are identified, Theorem 3 will remain valid when we replace a cress with a cresst.

6. The general coset representative systems. Let $G = \Pi^*(A_i;\ U_{kj} = U_{jk})$ be presented so that the α_i generating symbols include a set of generating symbols for each U_{ij} contained in A_i. Then the defining relators arising from an edge may be written $u_{ji}u_{ij}^{-1}$ where $\theta_{ji}(u_{ji}) = u_{ij}$ and u_{ji}, u_{ij} are among the α_j and α_i generating symbols respectively. Let H be a subgroup of G and to each pair A_i, U_{ij} associate a right coset representative function α_{ij}- for $G \bmod H$. We let C be the collection of the α_{ij}-. Thus, C is a collection of representative functions and when we wish to refer to the collection of all the representatives arising from these functions we use the notation $[C]$.

Assign a level function λ to G having exactly one vertex of level 0, and let G_n denote the subtree product generated by the vertices of level $\leqslant n$. Then $H_n = H \cap G_n$ is a subgroup of G_n. If Γ is the tree associated with G and Γ_n is the subtree associated with G_n then for each pair A_i, U_{ij} from Γ_n associate a right coset representative function $\alpha_{ij}(n)$- for $G_n \bmod H_n$. The collection of the $\alpha_{ij}(n)$- is called $C(n)$.

DEFINITION. C is regular if whenever the α_i generating symbols are deleted completely from the ends of the α_{ij}-representatives a double coset representative system for $G \bmod (H, A_i)$ results and is the same for each j.

DEFINITION. C is an ascending enlarged Schreier system if (i) when the G_n generating symbols are deleted completely from the ends of the α_{ij}-representatives, where $\lambda(A_i) \leqslant n$, a double coset representative system for $G \bmod (H, G_n)$ results and is the same for each such α_{ij}-representative system, (ii) the $\alpha_{ij}(n)$-representatives corresponding to all pairs of neighboring vertices A_j, A_i, where $\lambda(A_j) = n = \lambda(A_i) + 1$, form a cresst for $G_n \bmod H_n$ when G_n is realized as a tree product with amalgamations from the single vertex G_{n-1}.

We now turn to the notion of a compatible regular enlarged Schreier system for trees (cresst).

DEFINITION. The collection $\{\alpha_{ij}\text{-}\}$ is a cresst for $G \bmod H$ provided:

(1) The representatives form an ascending regular enlarged Schreier system for $G \bmod H$.

(2) When the u_{ij} symbols are deleted completely from the ends of the α_{ij}-representatives a double coset representative system for $G \bmod (H, U_{ij})$ results.

(3) An α_{ij}-representatives does not end in a u_{ji} symbol.

(4) If K is both a u_{ij}- and u_{ji}-double coset representative then $KP(u_{ij})$ is an α_{ij}-representative iff $KP(u_{ji})$ is an α_{ji}-representative.

(5) If neighboring vertices A_j and A_i have level n and $n-1$ respectively then an α_j-double coset representative ends in an α_i symbol or is both an α_j- and α_i-double coset representative.

THEOREM 4. *There is a cresst for $G \bmod H$ on the given generators.*

PROOF. We first prove by induction on n that if K is any subgroup of G_n then there is a cresst for $G_n \bmod K$ on the given generators for G_n.

The case $n = 1$ has been dealt with. Assume that a cresst can be constructed for $G_{n-1} \bmod J$ on the given generators for G_{n-1} where J is any subgroup of G_{n-1}.

Contract G_{n-1} to a vertex of G_n so that G_n is realized as a tree product with amalgamations from the single vertex G_{n-1}. Now construct a double coset representative system $\{D_r\}$ for $G_n \bmod (K, G_{n-1})$ as we constructed a minimal double coset representative system for $G \bmod (H, B)$ in the case $n = 1$. Each D_r

must end in a generating symbol from a vertex of level n which is not a u-symbol for an amalgamation with a vertex of level $n-1$.

By induction there is a cresst $C(n-1, D_r)$ for G_{n-1} mod $G_{n-1} \cap D_r^{-1}KD_r$ for each D_r, and if its α_{ij}-representatives have the form $D_{ri}E_{ij}P(u_{ij})$ then $\{D_rD_{ri}\}$ is a double coset representative system for G_n mod (K, A_i). Moreover, the E_{ij} are from a special Schreier system for A_i mod $A_i \cap D_{ri}^{-1}D_r^{-1}KD_rD_{ri}$ with respect to U_{ij}. To each D_r adjoin the representatives from the α_{ij}-representative systems of $C(n-1, D_r)$. This gives the $\alpha_{ij}(n)$-representatives for G_n mod K corresponding to the vertices of Γ_{n-1}.

Suppose now that A_j and A_i are neighboring vertices of level n and $n-1$ respectively. If $\{D_{ri}\}$ is the set of double coset representatives above for G_{n-1} mod $(G_{n-1} \cap D_r^{-1}KD_r, A_i)$ and if $\{E_{ij}P(u_{ij})\}$ is a special Schreier system for A_i mod $A_i \cap D_{ri}^{-1}D_r^{-1}KD_rD_{ri}$ with respect to U_{ij} then select $\{D_rD_{ri}E_{ij}P(u_{ij})\}$ as the $\alpha_{ij}(n)$-representatives. To complete this construction we go back to realizing G_n as having amalgamations from the single vertex G_{n-1}. Since $\{D_{ri}E_{ij}P(u_{ij})\}$ is a special Schreier system for G_{n-1} mod $G_{n-1} \cap D_r^{-1}KD_r$ with respect to U_{ij} we may take the $\alpha_{ij}(n)$-representatives $\{D_rD_{ri}E_{ij}P(u_{ij})\}$ as the $(G_{n-1})_j$-representatives. The length of each $(G_{n-1})_j$-representative equals the length of the corresponding D_r since $D_{ri}E_{ij}P(u_{ij})$ is a word in G_{n-1} generating symbols and consequently has (extremal) length 0. To obtain the double coset representative for KWA_j of length $r = l(W)$ notice that W is in KWG_{n-1} so the $(G_{n-1})_j$-representative of W has been defined and has the form $D_rD_{ri}E_{ij}P(u_{ij})$. Let $D_j = D_rD_{ri}E_{ij}$ be the (minimal) double coset representative for KWA_j and adjoin to D_j a special Schreier system for A_j mod $A_j \cap D_j^{-1}KD_j$ with respect to U_{ji} to get the α_{ji}-representatives for the cosets of K in KWA_j.

We show that all the representative systems form a cresst $C(n)$ for G_n mod K. Clearly when the G_{n-1} symbols are deleted completely from the ends of the α_{ij}-representatives, where $\lambda(A_i) \leq n-1$, a double coset representative system for G_n mod (K, G_{n-1}) results and is the same for each such α_{ij}-representative system. By induction $C(n-1, D_r)$ satisfies the ascending property, and since the $\alpha_{ij}(n)$-representatives are formed by adjoining the α_{ij}-representatives of each $C(n-1, D_r)$ to D_r it follows that $C(n)$ satisfies property (1) of a cresst. Properties (2), (3) and (5) are immediate from construction and (4) is easily handled.

Let H be any subgroup of G. We know there is a cresst $C(n)$ for G_n mod H_n for each n and $[C(n)] \subseteq [C(n+1)]$. Then $[C] = \bigcup[C(n)]$ determines a cresst C for G mod H where the α_{ij}-representatives are the ascending union of the $\alpha_{ij}(n)$-representatives, $n \geq \lambda(A_i)$.

(To check that C is a cresst note that $HWA_i = \bigcup(H_nWA_i)$ and $HWG_k = \bigcup(H_nWG_k)$.)

LEMMA 2. *If C is a cresst for G* mod H *then we must have* $[C] = \bigcup [C(n)]$ *where $C(n)$ is a cresst for G_n* mod H_n, $[C(n-1)] \subseteq [C(n)]$ *and the α_{ij}-representatives of C are the ascending union of the $\alpha_{ij}(n)$-, $n \geqslant \lambda(A_i)$.*

7. **The general subgroup theorem.** Let G be a tree product presented as in §6. We will investigate the structure of a subgroup H of the tree product G by examining the ascending union of the H_n.

Let a cresst C be given for G mod H. If neighboring vertices A_i and A_j of G have level $n-1$ and n respectively then the symbol $t_{j,D_jE_{ij}}$ will stand for $D_jE_{ji}[^{\alpha ij}(D_jE_{ji})]^{-1}$, where D_jE_{ji} is an α_{ji}-representative from C which is neither an α_i- nor α_j-double coset representative. The set of all such t symbols arising from C and the neighboring vertices of G is denoted $\{t_{j,D_jE_{ji}}\}$.

THEOREM 5. *Let H be a subgroup of G and $\{D_iE_{ij}P(u_{ij})\}$ the α_{ij}-representatives in a cresst C for G* mod H. *Then H is an HNN group whose free part is freely generated by $\{t_{j,D_jE_{ji}}\}$. Moreover, the base of H is a tree product whose vertices are all the subgroups $A_{iH}^{D_i}$ and:*

(1) *$A_{iH}^{D_i}$ and $A_{jH}^{D_j}$ are neighbors iff A_i and A_j are neighbors in G and $D_i = D_j$ or $D_i = D_jE_{ji}$ or $D_j = D_jE_{ij}$;*

(2) *The edge between neighbors $A_{iH}^{D_i}$ and $A_{jH}^{D_j}$ is U_{ijH}^{D} where $D = D_i = D_j$ or D is the longer of D_i and D_j in syllable length;*

(3) *The relations involving the free part of H and the associated subgroups are*

$$t_{j,D_jE_{ji}} U_{ijH}^{D_iE_{ij}} t_{j,D_jE_{ji}}^{-1} = U_{ijH}^{D_jE_{ji}},$$

where $D_iE_{ij} = {}^{u_{ij}}|D_jE_{ji}|$.

PROOF. Let K be a subgroup of G_n. We prove by induction on n that if $C(n)$ is any cresst for G_n mod K then K has the structure described by this theorem. It will be obvious that $K_{n-1} = G_{n-1} \cap K$ is naturally contained in K_n for $n \geqslant 1$. From this and the fact that $[C] = \bigcup [C(n)]$ it will follow that (see [11, p. 33, Problem 18]) $H = \bigcup H_n$ has the desired structure.

The case $n = 1$ has been dealt with, so assume $n > 1$. Clearly $C(n-1)$ is a cresst for G_{n-1} mod K_{n-1} so by inductive hypothesis the subgroup K_{n-1} of G_{n-1} has the structure described by this theorem. Since G_n is a tree product with amalgamations from the single vertex G_{n-1} and since the $\alpha_{ij}(n)$- and $(G_{n-1})_j$-representatives coincide we may apply the theorem proved for the case $n = 1$. Thus, K is an HNN group whose free part is freely generated by $\{t_{j,D_jE_{ji}}\}$ where A_j has level n in G_n. The base of K is a tree product with vertices all the $A_{jK}^{D_j}$ and $G_{n-1_K}^{D_r}$ where again A_j has level n in G_n and $\{D_r\}$ ranges over a double coset representative system for G_n mod (K, G_{n-1}). The $(G_{n-1})_j$-representatives must have the form $D_r D_{ri} E_{ij} P(u_{ij})$ where $\{D_{ri} E_{ij} P(u_{ij})\}$ is a special Schreier system for G_{n-1} mod $G_{n-1} \cap D_r^{-1} K D_r$ with respect to U_{ij}. It follows that the edge relations in this presentation are

(i) $$U_{ji_K}^{D_j E_{ji}} = U_{ij_K}^{D_r D_{ri} E_{ij}},$$

where $U_{ij_K}^{D_j E_{ji}}$ is a subgroup of $A_{j_K}^{D_j}$, $U_{ij_K}^{D_r D_{ri} E_{ij}}$ is a subgroup of $G_{n-1_K}^{D_r}$ and $D_j E_{ji} = D_r D_{ri} E_{ij}$. This latter equation implies that at least one of E_{ji} and E_{ij} is 1. Moreover, the relations involving the associated subgroups and the free part of K are

(ii) $$t_{j,D_j E_{ji}} U_{ij_K}^{D_r D_{ri} E_{ij}} t_{j,D_j E_{ji}}^{-1} = U_{ij_K}^{D_j E_{ji}},$$

where $D_r D_{ri} E_{ij} = {}^{u_{ij}}|D_j E_{ji}|$ and the associated subgroups are contained in $G_{n-1_K}^{D_r}$ or $A_{j_K}^{D_j}$.

We examine the vertices $G_{n-1_K}^{D_r}$ to complete the proof. Clearly $G_{n-1_K}^{D_r} = D_r (G_{n-1} \cap D_r^{-1} K D_r) D_r^{-1}$ and $\{D_{ri}\}$ is a double coset representative system for G_{n-1} mod $(G_{n-1} \cap D_r^{-1} K D_r, A_i)$ from a cresst for G_{n-1} mod $G_{n-1} \cap D_r^{-1} K D_r$. By induction

$$G_{n-1} \cap D_r^{-1} K D_r = gp(A_{i_{G_{n-1} \cap D_r^{-1} K D_r}}^{D_{ri}}, t_{i,D_{ri} E_{ij}})$$

where A_i is a vertex of G_{n-1} and $G_{n-1} \cap D_r^{-1} K D_r$ has defining relations as described by this theorem on these generators. Since D_{ri} is in G_{n-1} and A_i is contained in G_{n-1}

$$A_{i_{G_{n-1} \cap D_r^{-1} K D_r}}^{D_{ri}} = A_{i_{D_r^{-1} K D_r}}^{D_{ri}},$$

so the vertex $G_{n-1_K}^{D_r}$ is generated by the $A_{i_K}^{D_r D_{ri}}$ and the

$$D_r (D_{ri} E_{ij})^{\alpha_{ji}} (D_{ri} E_{ij})^{-1} D_r^{-1} = D_r D_{ri} E_{ij} [^{\alpha_{ji}} (D_r D_{ri} E_{ij})]^{-1},$$

for which we use the symbol $t_{i,D_r D_{ri} E_{ij}}$. Moreover, conjugating by D_r we see that $G_{n-1_K}^{D_r}$ has edge and associated subgroups of the form $U_{ij_K}^{D_r D_{ri} E_{ij}}$ and that the subgroups arising in relations (i) and (ii) are contained in some $A_{i_K}^{D_r E_{ri}}$ (or, as before, in $A_{j_K}^{D_j}$).

Since $G_{n-1_K}^{D_r}$ is naturally contained in K we may replace it by the generators and relations described above. This shows K has generators and defining relations as described by this theorem. It is easy to see that the t symbols freely generate a free subgroup of K, and to establish the HNN structure notice that an isomorphism between subgroups in our original presentation for K may be regarded as an isomorphism between subgroups of vertices in the latter presentation. To see that Γ_n is a tree notice that the graph Γ_1 from the base of our original presentation for K is a tree with vertices $A_{jK}^{D_j}$ and $G_{n-1_K}^{D_r}$ and Γ_n is obtained by replacing each of these latter vertices with a tree (using the edges from (i)).

COROLLARY 5.1. *The free part of H is a retract of H with normal complement N, the normal subgroup generated by the tree product of the vertices, S. If H is generated by its intersections with conjugates of vertices then $H = S$.*

COROLLARY 5.2. *If H intersects the conjugates of the edges trivially then H is the free product of its free part F and S, and S is the free product of its vertices. If H intersects the conjugates of the vertices trivially then $H = F$.*

COROLLARY 5.3. *Let $T_{D_jE_{ji}} = gp(t_{j,D_jE_{ji}}, S)$. Then $T_{D_jE_{ji}}$ is the HNN group*

$$\langle t_{j,D_jE_{ji}}, S; \operatorname{rel}(S), t_{j,D_jE_{ji}} U_{ijH}^{D_iE_{ij}} t_{j,D_jE_{ji}}^{-1} = U_{ijH}^{D_jE_{ji}}\rangle$$

and $H = \Pi^*(T_{D_jE_{ji}}; S)$. If

$$S_{D_jE_{ji}} = gp(U_{ijH}^{D_iE_{ij}}, U_{ijH}^{D_jE_{ji}})$$

then $T_{D_jE_{ji}}$ is the free product of

$$T'_{D_jE_{ji}} = \langle t_{j,D_jE_{ji}}, S_{D_jE_{ji}}; \operatorname{rel}(S_{D_jE_{ji}}), t_{j,D_jE_{ji}} U_{ijH}^{D_iE_{ij}} t_{j,D_jE_{ji}}^{-1} = U_{ijH}^{D_jE_{ji}}\rangle$$

(which is an HNN group) and S with $S_{D_jE_{ji}}$ amalgamated.

All three of these corollaries are proved as in [9].

8. **A level function for the base.** Let $G = \Pi^*(A_i; U_{jk} = U_{kj})$ with B the only vertex of level 0. If H is any subgroup of G presented using Theorem 5 then we will develop a level function for the base S of H such that given any vertex X_H^D of S we may calculate its level directly, and B_H will be the only vertex of level 0.

If W_1 and W_2 are α_i and α_j syllables respectively then let $j(W_1, W_2)$ be the number of edges of G in the simple path from A_i to A_j. If W is any word in the generating symbols of G and $W = W_1 W_2 \ldots W_n$ where $W_1, W_2, \ldots, W_n$ are the syllables of W then we define

$$j(W) = \sum_{i=1}^{n-1} j(W_i, W_{i+1})$$

and call $j(W)$ the sum of the jumps in W.

DEFINITION. If X_H^D is a vertex of S and V_1 and V_2 are β and x syllables respectively then $\lambda(X_H^D) = j(V_1 D V_2)$.

THEOREM 6. *λ defines a level function on S with B_H the only vertex of level* 0.

PROOF. By the syllable length of a vertex X_H^D we mean the syllable length of D. We first show that for any n the vertices of syllable length $\leqslant n$ form a complete set of vertices for a subtree of S. If $n = 0$ the vertices $\{A_{i_H}\}$ and the edges between them form a subtree isomorphic to the graph of G. Assume that the vertices of syllable length $\leqslant r$ form a complete set of vertices for a subtree of S and let X_H^D be any vertex of syllable length $r + 1$ with $X_H^D, X_{1_H}^{D_1}, \ldots, X_{n_H}^{D_n}$, $Y_H^{D'}$ the consecutive vertices from X_H^D to the nearest vertex of syllable length $\leqslant r$, $Y_H^{D'}$. By Theorem 5 $D_n = D'E_y$ and we claim that $D = D_1 = \ldots = D_n = D'E_y$. For suppose otherwise. Since each D_i must have syllable length $\geqslant r + 1$ we have that some D_i has maximal syllable length $> r + 1$. If j and k are the minimal and maximal subscripts such that $D_j = D_i = D_k$ then the consecutive vertices above are $X_H^D, X_H^{D_1}, \ldots, X_{j-1_H}^{D_{j-1}}, X_{j_H}^{D_i}, \ldots, X_{k_H}^{D_i}, X_{K+1_H}^{D_{k+1}}, \ldots, Y_H^{D'}$, where $j \geqslant 1$ and $k < n$. Then $D_i = D_{j-1}E_{j-1} = D_{k+1}E_{k+1}$ with $E_{j-1} \neq 1 \neq E_{k+1}$ so E_{j-1} and E_{k+1} are the same syllable. This means $X_{j-1} = X_{k+1}$, $D_{j-1} = D_{k+1}$ and the vertex $X_{j-1_H}^{D_{j-1}}$ appears twice in S, a contradiction. It follows that the vertices of syllable length $\leqslant r + 1$ form a complete set of vertices for a subtree of S.

We call the subtree of vertices of syllable length $\leqslant n$ T_n and prove by induction on n that λ is a level function on S.

Clearly $\lambda(A_{i_H}) = j(V_1 1 V_2)$ where V_1 is a β syllable and V_2 an α_i syllable assigns to A_{i_H} the number of edges from it to B_H. This makes λ a level function on T_0. Suppose λ defines a level function on T_r and let $A_{i_H}^D$ have syllable length $r + 1$. If $A_{i_H}^D = X_{1_H}^D, X_{2_H}^D, \ldots, X_{n_H}^D, Y_H^{D_y}$ are the vertices from $A_{i_H}^D$ to the nearest vertex $Y_H^{D_y}$ of T_r then $X_i \neq X_j$ for $i \neq j$, and $D = D_y E_y$ implies E_y cannot be any X_i syllable so $X_i \neq Y$ for each i. Since $X_1, X_2, \ldots, X_n, Y$ are distinct consecutive vertices in the graph of G it follows that they determine a simple path there. Thus, if V_1, V_2 and V_3 are β, x_1, and y syllables respectively then $j(V_1 D V_2) = j(V_1 D_y E_y V_2) = j(V_1 D E_y) + j(E_y V_2) = j(V_1 D V_3) + j(E_y V_2)$. By induction $\lambda(Y_H^D) = j(V_1 D_y V_3)$ gives the number of edges from Y_H^D to B_H and we just showed $j(E_y V_2)$ is the number of edges from $A_{i_H}^D$ to Y_H^D. This proves λ is a level function on T_{r+1}.

We use λ_H to avoid confusion between the level functions for G and the base S of H.

COROLLARY 6.1. *If $\lambda(\begin{smallmatrix} i \\ H \end{smallmatrix})(X_H^D) = j(V_i D V_2)$ where V_i is an α_i syllable and*

V_2 an x syllable then $\lambda_H^{(i)}$ is a level function on S with A_{i_H} the unique vertex of level 0.

COROLLARY 6.2. *If $\lambda'_H(X_H^D) = j(DV)$ with V an x syllable then λ'_H is a level function on S with the vertices $\{A_{i_H}\}$ forming the set of vertices for the subtree of level* 0.

λ'_H reduces to the level function used in [9] for the base of a subgroup of $G = (A * B; U)$.

COROLLARY 6.3. *If X_H^D has greater syllable length than its neighboring vertex Y_H^D then $\lambda_H(X_H^D) = \lambda_H(Y_H^D y) + 1$.*

9. **The rank of the free part.** It is shown in [9] that if $G = (A * B; U)$ then the rank of the free part of a subgroup H of G is, when presented as described by the Karrass-Solitar theorem, $G : (H, U) - G : (H, A) - G : (H, B) + 1$. If H is a subgroup of a general tree product presented as described by Theorem 5 then it is more difficult to express the rank of its free part because the cresst is complicated and because we may have infinite paths in which each vertex X_H^D has the same superscript.

Assign a level function to G having exactly one vertex of level 0.

THEOREM 7. *Let $G = \Pi^*(A_i; U_{jk} = U_{kj})$ have finitely many levels and let H be a subgroup of G presented using Theorem 5. Then the free part of H has rank $\Sigma G : (H, U_{ij}) - \Sigma G : (H, A_i) + 1$, where we use only one of U_{ij}, U_{ji}.*

Let F denote the free part of H. We will show that the number of t symbols in F corresponding to all the edges between vertices of level n and $n - 1$ is

$$\text{(1)} \qquad \sum G : (H, U_{ij}) - \sum G: (H, A_i) - G(H, G_{n-1}) + G : (H, G_n),$$

where A_i ranges over the vertices of level n and U_{ij} is contained in some A_i.

Suppose that $D_iE_{ij}{}^{\alpha_{ji}}(D_iE_{ij})^{-1} \approx 1$. Then since D_iE_{ij} is a u_{ij}-double coset representative from a cresst for $G \bmod H$ we must have $D_iE_{ij} = D_jE_{ji}$. Now $E_{ij} = 1$ if and only if D_i is an α_i-double coset representative, while if $E_{ij} \neq 1$ then D_j is an α_j-double coset representative ending in an α_i symbol in which case D_j must be a double coset representative for $G \bmod (H, G_{n-1})$. Let X_i be the set of (H, G_{n-1}) representatives which end in an α_i symbol. Then the members of X_i as well as the α_i-double coset representatives are u_{ij}-double coset representatives, so the number of t symbols contributed to F by the U_{ij} must be

$$\sum G : (H, U_{ij}) - \sum G : (H, A_i) - \sum |X_i|.$$

Further, the (H, G_{n-1}) representatives which result from deleting the G_{n-1} symbols end in a symbol from a vertex of level n or are both (H, G_{n-1}) and (H, G_n) representatives. This means $G:(H, G_{n-1}) = \Sigma|X_i| + G:(H, G_n)$ and (1) follows.

If we now sum the expressions obtained in (1) we see that the number of t symbols corresponding to all the edges of G_k is $\Sigma G:(H, U_{ij}) - \Sigma G:(H, A_i) + G:(H, G_k)$. Since $G = G_n$ for some n, this theorem is proved.

COROLLARY 7.1. *If $G = \Pi^*(A_i; U_{jk} = U_{kj})$ has finitely many levels and N is a normal subgroup whose free part has finite rank when N is presented as described by Theorem 5 then this rank is*

$$(G:N)\left[\sum|N \cap U_{ij}|/|U_{ij}| - \sum|N \cap A_i|/|A_i|\right] + 1,$$

where we count only one of U_{ij}, U_{ji}.

PROOF. If D is a double coset representative for G mod (N, X) then $(G:N) = \Sigma_D(X:D^{-1}ND \cap X) = [G:(N, X)](X:N \cap X)$ since N is normal.

10. Indecomposable subgroups and subgroups which satisfy an identity.

THEOREM 8. *Let $G = \Pi^*(A_i; U_{jk} = U_{kj})$ and suppose H is an indecomposable subgroup of G. Then H is one of the following:*

(1) *A subgroup of a conjugate of a vertex;*

(2) *A countable ascending union $U_{iH}^{D_i}$, where each U_i is some U_{jk} and D_i is an α_k-double coset representative. Moreover, either*

(a) *$D_i = D_{i+1}$ in which case U_i and U_{i+1} are consecutive edges of G, or*

(b) *D_i is obtained by deleting the last syllable of D_{i+1} in which case U_i and U_{i+1} are the same or are consecutive edges of G;*

(3) *An HNN group of the form $\langle t, S';\ \mathrm{rel}(S'),\ tU_H^{\delta}t^{-1} = U_H^{\delta'}\rangle$, where U is some U_{kj}, δ and δ' are distinct u_{jk}- and u_{kj}-double coset representatives for the same (H, U) coset, $S' = gp(U_H^{\delta}, U_H^{\delta'})$ and $t = \delta' P \delta^{-1}$ with P in U.*

PROOF. Present H using Theorem 5. If the free part of H is nontrivial then we use Corollary 5.3 together with the argument of Theorem 6 of [9] to show that H must be an HNN group in (3) above.

Suppose then that the free part of H is trivial so that $H = S$. By Theorem 2 of [9] G equals one of its vertices or is an ascending union of edges from a simple path containing one vertex $A_{iH}^{D_i}$ of each λ_H level $i \geqslant 0$. Thus, if H is not in a conjugate of a vertex of G it is an ascending union of $U_{iH}^{D_i}$. If $D_i = D_{i+1}$ then U_i and U_{i+1} are distinct consecutive edges of G by Theorem 5. If $D_i \neq D_{i+1}$ then by Corollary 6.3 D_i and D_{i+1} must be as in (2b) above.

COROLLARY 8.1. *If the edges of G are all countable then any uncountable indecomposable subgroup H is contained in a conjugate of a vertex.*

COROLLARY 8.2. *An abelian subgroup of G has the form* (1), (2), *or is the direct product of an infinite cyclic group and a subgroup of a conjugate of an edge.*

THEOREM 9. *If $G = \Pi^*(A_i; U_{jk} = U_{kj})$ and H is a subgroup of G which satisfies a nontrivial law then H is one of the following:*

(1) *a subgroup of a conjugate of a vertex;*

(2) *an ascending union as in* (2) *of Theorem* 8;

(3) *an amalgamated product $(A_{i_H}^{D_i} * A_{j_H}^{D_j}; U_H^D)$, where U_H^D has index two in both factors and $D = D_i$ or $D = D_j$;*

(4) *an* HNN *group as in* (3) *of Theorem* 8, *where $U_H^\delta < U_H^{\delta'}$ or $U_H^\delta < U_H^{\delta'}$ and H is an infinite cyclic extension of U_H^δ, $U_H^{\delta'}$ or of a proper ascending union of the form $\bigcup U_H^{t^k\delta}$ or $\bigcup H_U^{t^{-k}\delta}$, $k \in \{0, 1, \ldots\}$.*

PROOF. Present H using Theorem 5. If the free part of H has rank $\geqslant 2$ then H cannot satisfy a nontrivial law. If the free part of H is infinite cyclic we use Corollary 5.3 together with the argument of Theorem 7 of [9] to see that H must be as in (4) above. If $H = S$ then applying Theorem 3 of [9] and using λ_H as our level function we see that H is as in (1), (2), or (3).

11. The cartesian subgroup of a tree product whose edges are contained in the centers of their vertices.

DEFINITION. Let $G = \Pi^*(A_i; U_{jk} = U_{kj})$ be a tree product with each U_{jk} contained in the center of A_j (and A_k). Then the group obtained by adjoining all the relations $a_s a_r a_s^{-1} a_r^{-1} = 1$, where $a_s \in A_s$, $a_r \in A_r$ and $s \neq r$, is called the direct tree product of the A_i amalgamating U_{jk} and U_{kj} (under θ_{jk}).

By analogy with tree products, it is clear what is meant by a direct tree product with a single amalgamated subgroup (for which we use the notation $\Pi(A_i; U)$) or by a direct tree product with amalgamations from a single vertex.

It is easy to see that if $G = \Pi(A_i; U_{jk} = U_{kj})$ and A_i is naturally contained in G for each i then U_{ij} must be contained in the center of A_i (and A_j).

THEOREM 10. *If $G = \Pi(A_i; U_{jk} = U_{kj})$ is a direct tree product then each A_i is naturally contained in G. Moreover, the direct tree product of a subtree is contained in G in the natural way.*

PROOF. Suppose first that G is a direct tree product with amalgamations from the single vertex A_0. If U is the subgroup of A_0 generated by all the U_{0i} then clearly U is contained in the center of A_0. Corresponding to each i we may form the direct tree product $(A_i \times U; U_{i0} = U_{0i})$. Consider

$$G_0 = \prod[(A_i \times U; U_{i0} = U_{0i}); U],$$

where $U_{00} = U$ and the natural copies of U are set identically equal. Since we may add relations to G_0 which set *every* natural copy of U from a vertex of G_0 identically equal, G_0 is the generalized direct product of the factors $(A_i \times U; U_{i0} = U_{0i})$ with the single amalgamated subgroup U from the center of each factor. It follows that $(A_i \times U; U_{i0} = U_{0i})$ is naturally contained in G_0 and similarly A_i is naturally contained in $(A_i \times U; U_{i0} = U_{0i})$. This shows A_i is naturally contained in G_0. Now G is isomorphic to G_0 under the mapping $\Phi: G \longrightarrow G_0$ induced by $a_i \longrightarrow a_i$. This implies that A_i is naturally contained in G.

Assign a level function to the graph of G having exactly one vertex of level 0 and let G_n be the direct tree product of the vertices of level $\leqslant n$. The presentation for G_n is contained in the presentations for G_{n+1} and G. We prove by induction that each vertex of G_n is naturally contained in G_n and G_n is naturally contained in G_{n+1}.

If $n = 0$ we are done, so assume that the vertices of G_n are naturally contained in G_n. Let $U_{kj} = U_{jk}$ correspond to the edge of G_{n+1} between vertices A_k and A_j of level n and $n - 1$ respectively. Then U_{jk} is contained in the center of A_j, and hence, in the center of G_n. It follows that G_{n+1} is a direct tree product with amalgamations from the single vertex G_n, so applying the first part of this proof G_n and the vertices of level $n + 1$ are naturally contained in G_{n+1}. It is easy to see then that G_{n+1} naturally contains all its vertices. Moreover (by [11, p. 33, Problem 18]) G_n is naturally contained in G, and, hence, A_i is naturally contained in G.

For the last part of this theorem contract the desired subtree to a vertex, assign it level 0, and apply the above argument.

THEOREM 11. *If $G = \Pi^*(A_i; U_{jk} = U_{kj})$ where each U_{jk} is contained in the center of its two vertices then $C(G)$ is a free group.*

PROOF. If $H = \Pi(A_i; U_{jk} = U_{kj})$ then by the previous theorem the mapping $\Phi: G \longrightarrow H$ determined by $a_i \longrightarrow a_i$ is a homomorphism which defines the identity mapping on each A_i. This means that $A_i \cap \ker \Phi = 1$, so if we present $\ker \Phi$ using Theorem 5 it follows easily that $\ker \Phi$ is a free group. From the presentations it is clear that $\ker \Phi = C(G)$.

In [1], Anshel and Prener prove that if $G = \Pi_{i=1}^{r}{}^*G_i$ then G' is free of rank $rn - n\Sigma 1/q_i - (n - 1)$, where $q_i = |G_i/G_i'|$ and $n = \Pi_{i=1}^{r} q_i$. They emphasize that when the G_i are finite abelian groups the rank of G' depends only on r and the orders of the factors.

LEMMA 4. *Suppose $G = \Pi_{i=1}^{r}(A_i;\ U_{jk} = U_{kj})$ where each A_i is finite and each edge is contained in the center of its two vertices. Then $|G| = (\Pi |A_i|)/(\Pi |U_{jk}|)$, where we count only one of U_{kj}, U_{jk}.*

PROOF: We use induction on r. If $r = 2$ it is easy to see that each element of $(A_1 \times A_2;\ U_{12})$ has a unique expression of the form ua_1a_2, where $u \in U_{12}$ and a_1 and a_2 are from right tranversals for A_1 mod U_{12} and A_2 mod U_{12} respectively. Then $|G|$ is as claimed. To establish the inductive step treat G as having two factors, one of them an extremal vertex of G.

If $G = \Pi_{i=1}^{r}{}^{*}(A_i;\ U_{jk} = U_{kj})$ with each A_i finite then let $n = (\Pi |A_i|)/(\Pi |U_{kj}|)$ where we count only one of U_{jk}, U_{kj}.

THEOREM 12. *If $G = \Pi_{i=1}^{r}{}^{*}(A_i;\ U_{jk} = U_{kj})$ where each A_i is finite and the edges are contained in the centers of their two vertices then $C(G)$ is free of rank*

$$n\sum 1/|U_{jk}| - n\sum 1/|A_i| + 1,$$

where we count only one of U_{jk}, U_{kj}. If the A_i are abelian this is also the rank of G' which will then be free. If each $U_{jk} = \{1\}$ this expression reduces to

$$rn - n\sum 1/|A_i| - (n-1).$$

In any case the rank of $C(G)$ depends only on r and the orders of the vertices and edges.

PROOF. $(G : C(G)) = n$ by Lemma 4 so all we need do is apply Corollary 7.1.

12. **Some groups which are finite extensions of free groups.** In this section ρ is the standard embedding of a group G into S_G by right multiplications.

THEOREM 13. *Let $G = \pi^{*}(A_i;\ U_{jk} = U_{kj})$ where the edges are finitely generated subgroups of finite index in both their vertices and some edge is a proper subgroup of both its vertices. Then G is a finite extension of a free group if and only if the orders of the A_i are uniformly bounded.*

PROOF. Suppose first that the orders of the vertices are uniformly bounded and $\{X_1, \ldots, X_n\}$ is a transversal for the partition of the vertices into isomorphism classes. We set $K = X_1 \times X_2 \times \ldots \times X_n$ and construct a homomorphism $\psi: G \longrightarrow S_K$ which embeds each vertex. Then $\ker \psi \cap gA_ig^{-1} = g(\ker \psi \cap A_i)g^{-1} = 1$ and by Corollary 5.2 $\ker \psi$ is a free group. Since S_K is finite $(G_i : \ker \psi)$ is finite.

Let $\widetilde{X}_i$ be the natural copy of X_i in K. Assign a level function to G having exactly one vertex of level zero and call that vertex A_0. We define ψ on the vertices of G so that the following two properties hold:

(i) If A is any vertex and A is isomorphic to X_s then ψ is defined on A as the composite of isomorphisms $A \longrightarrow \widetilde{X}_s \longrightarrow \rho(\widetilde{X}_s) \longrightarrow \gamma^{-1}\rho(\widetilde{X}_s)\gamma$, where $\gamma \in S_k$;

(ii) If $u = \theta_{jk}u$ is any relation corresponding to an amalgamation then $\psi(u) = \psi(\theta_{jk}u)$.

Define ψ on A_0 as the composite of isomorphisms $A_0 \longrightarrow \widetilde{X}_i \longrightarrow \rho(\widetilde{X}_i)$ where A_0 is isomorphic to X_i. This defines ψ on the vertices of level $\leqslant 0$. Suppose ψ has been defined on each vertex of level $< r$, $r > 0$, so that (i) and (ii) hold. Let A_i be a vertex of level r which is joined by an edge to the vertex A_k of level $r-1$. By inductive hypothesis ψ is defined on A_k by composing isomorphisms of the form $A_k \longrightarrow \widetilde{X}_s \longrightarrow \rho(\widetilde{X}_s) \longrightarrow \gamma^{-1}\rho(\widetilde{X}_s)\gamma$. If U_{ik} and U_{ki} are the subgroups of A_i and A_k joined by θ_{ki}: $U_{ki} \longrightarrow U_{ki}$ and A_i is isomorphic to X_j then the copies $\widetilde{U}_{ki}$, $\widetilde{U}_{ik}$ in $\widetilde{X}_s$, $\widetilde{X}_j$ are isomorphic under the copy $\widetilde{\theta}_{ki}$ of θ_{ki} where $\widetilde{\theta}_{ki}$ pairs the images in $\widetilde{X}_s$ and $\widetilde{X}_j$ of u and $\theta_{jk}u$. The image of $u \in U_{ki}$ in S_k is $\gamma^{-1}\rho(\widetilde{u})\gamma$, while $A_i \longrightarrow \widetilde{X}_j \longrightarrow \rho(\widetilde{X}_j)$ maps $\theta_{ki}u$ to $\rho(\widetilde{\theta}_{ki}\widetilde{u})$. It follows from a result due to P. Hall (see [12, p. 537]) that there is some π in S_k such that $(\pi\gamma^{-1})\rho(\widetilde{u})(\gamma\pi^{-1}) = \pi(\widetilde{\theta}_{ki}\widetilde{u})$, and we define ψ on A_i to be the composite of the isomorphisms

$$A_i \longrightarrow \widetilde{X}_j \longrightarrow \rho(\widetilde{X}_j) \longrightarrow \pi^{-1}\rho(\widetilde{X}_j)\pi.$$

Clearly $\psi(u) = \psi(\theta_{jk}u)$ for each u in U_{jk}.

By induction we have defined ψ on the generators of the tree product G in such a way that any defining relator of G maps to 1. It follows that ψ determines a homomorphism. Clearly ψ is embeds the vertices.

Conversely, if U_{ij} is a proper subgroup of both A_i and A_j then G a finite extension of a free group F implies $(A_i * A_j;\, U_{ij})$ is also a finite extension of a free group. By Karrass and Solitar [10] or Allenby and Gregorac [1], A_i is finite. If A_k is any other vertex of G there is a finite simple path from A_k to A_i in which each edge has finite index in both its vertices, so A_k must be finite. Further, if Φ: $G \longrightarrow G/F$ is the canonical map then since each vertex is finite it is clear that F free implies $\ker \Phi \cap A_i = 1$ for each A_i. Thus, Φ embeds A_i into the finite group G/F and it follows that only finitely many A_i may be nonisomorphic.

COROLLARY 13.1. *Let $G = \Pi^*(A_i;\, U_{jk} = U_{kj})$ where each A_i is periodic. Then every torsion-free subgroup of G is a free group and G is a finite extension of a torsion-free subgroup if and only if the orders of the A_i are uniformly bounded.*

PROOF. Let H be a torsion-free subgroup of G. Present H using Theorem 5. A vertex of the base S of H is of the form $DA_iD^{-1} \cap H = 1$ since DA_iD^{-1} is a periodic group. It follows that H is free. Moreover, if H is normal and has finite index in G then $|A_i| = |A_i/1| = |A_i/A_i \cap H|$ shows A_i must be finite. To complete this proof we may use the arguments of Theorem 13 to show that only finitely many A_i may be nonisomorphic and that G has a free subgroup of finite index when the A_i are finite and their orders are uniformly bounded.

EXAMPLE. Let us call any group of the form $(A * A; U)$ a generalized free square (see G. Baumslag, [3]) and $A *_U A *_U A *_U A *_U \ldots *_U A$ (n copies of A) a generalized free nth power of A. Then any generalized free nth power of a finite group has a free subgroup of finite index, even if n is infinite.

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